Local behavior of certain elliptic equations

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Abstract

The local behavior of some elliptic equations of the steady states of reaction diffusion equations is studies. It is well known that the competition between the produced rate, removal rate and diffusion coefficient affect the solution behavior. Such a phenomenon is called Turing instability. The purpose of this article is to discuss how the value of the interior point of the initial value tangles with these parameters when the competition of these parameters reach the balanced states. The mathematical meaning of balanced states will be specified in the article. It is interesting that the behavior of such a state coincide with positive conclusion of Lin-Ni conjecture.

Keywords: reaction diffusion, steady states, Turing system, local behavior

1 Introduction

The subject of this article is to study the local behavior of certain elliptic equations which is the steady states of reaction diffusion equations. It is well known that the competition between the parameters of the reaction diffusion equations affect the local behavior of the solution. Such a phenomenon is known as Turing instability [1]. The instability then derives the changing of pattern formation [2, 3, 4, 5, 6, 7, 8] which depends on the local behavior of the solution. Usually, the system of Turing system contains many parameters that shelter certain phenomenon form exploring. In fact, at certain value of parameters the sum of the solutions behave as a constant which we call it balanced states. It is interesting that the behavior of such a state coincide with positive conclusion of Lin-Ni conjecture [9]. The equation we study is as follow;

\[
\begin{align*}
  u_t &= D_u \Delta u + f(u, v) - au, & u \in \Omega, \\
  v_t &= D_v \Delta v + g(u, v) - bv + c,
\end{align*}
\]

with Neumann boundary condition

\[
\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0, \quad x \in \partial \Omega,
\]

and initial data

\[
u(0, \cdot) = u_0, \quad v(0, \cdot) = v_0.
\]
2 Pohozeav identity

For certain $f(u,v)$, $g(u,v)$ with right product rate $e$, removal rate $a$, $b$ and diffusion coefficient $D_u$, $D_v$ the steady state of system (1) may reduce to a singleton. To this end, we rescale variable and normalize the solution by letting $y = \frac{x}{\delta}$, $s = \frac{t}{\delta^2}$ and without ambiguity we still use variables $x$ and $t$ instead of $y$ and $s$ then (1) may rewrite as:

$$\Delta u + f(u,a,b) = 0. \quad (4)$$

To explore the local behavior of the solution, we apply Pohozeav identity locally. Without lose of generality, we assume 0 is an interior point of $\Omega$.

**Theorem 1.** If $u$ is a solution of (4) and if $u(0) = u_0 \geq \frac{\xi}{2}$ then the peak of solution occurs at the boundary.

**Proof:** Suppose $u$ has an interior local maximum, say at 0. By normalization as previous section, we may assume $u(0) = 1$. Thus for given $\epsilon > 0$ there exists a $\delta > 0$ such that $\nabla u \cdot x < 0$ for all $x \in B_\delta(0)$ and $u \geq 1 - \epsilon$. Let $B_\delta(0) = B_\delta$.

$$\int_{B_\delta} (\Delta u + f(u)) \cdot (\nabla u \cdot x) dx = 0,$n

$$\int_{B_\delta} \Delta u \cdot (\nabla u \cdot x) dx = \int_{\partial B_\delta} \frac{\partial u}{\partial n} \cdot (\nabla u \cdot x) ds - \int_{B_\delta} \nabla u \cdot \nabla (\nabla u \cdot x) dx,$n

On $\partial B_\delta(0)$, we have $x = \delta n$, thus

$$\int_{\partial B_\delta} \frac{\partial u}{\partial n} \cdot (\nabla u \cdot x) ds = \delta \int_{\partial B_\delta} \left( \frac{\partial u}{\partial n} \right)^2 dx.$n

$$\int_{B_\delta} \nabla u \cdot \nabla (\nabla u \cdot x) dx = \int_{B_\delta} \sum_i \left( u_{x_i} + \sum_j u_{x_i x_j} + u_{x_i x_i} \right) dx$$

$$= \int_{B_\delta} \frac{u}{2} |\nabla u|^2 \cdot x dx$$

$$= \int_{\partial B_\delta} \frac{u}{2} |\nabla u|^2 (x \cdot n) ds - \int_{B_\delta} \frac{N|\nabla u|^2}{2} dx$$

$$= \frac{\delta^2}{2} \int_{\partial B_\delta} \left( \frac{\partial u}{\partial n} \right)^2 ds - \int_{B_\delta} \frac{N|\nabla u|^2}{2} dx.$n

$$\int_{B_\delta} \Delta u \cdot (\nabla u \cdot x) dx = \delta \int_{\partial B_\delta} \left( \frac{\partial u}{\partial n} \right)^2 dx - \frac{\delta^3}{2} \int_{\partial B_\delta} \left( \frac{\partial u}{\partial n} \right)^2 ds + \int_{B_\delta} \frac{N|\nabla u|^2}{2} dx.$n

Let $F(u) = \int_0^u f(s) ds$ then

$$F(u) = \frac{u^{m+2}}{m+2} - \frac{\alpha u^{m+1}}{m+1} + \frac{\beta u^2}{2}.$n

$$\int_{B_\delta} f(u) \cdot (\nabla u \cdot x) dx = \int_{B_\delta} \nabla F(u) \cdot x dx$$

$$= \delta \int_{\partial B_\delta} F(u) ds - N \int_{B_\delta} F(u) dx \quad (5)$$

Thus,

$$0 = \int_{B_\delta} (\Delta u + f(u)) \cdot (\nabla u \cdot x) dx$$

$$= \delta \int_{\partial B_\delta} \left( \frac{\partial u}{\partial n} \right)^2 ds - \frac{\delta^3}{2} \left( \frac{\partial u}{\partial n} \right)^2 + F(u) ds + N \int_{B_\delta} \frac{|\nabla u|^2}{2} - F(u) dx. \quad (6)$$

Multiplying $u$ to the equation above and integrating over $B_\delta$ it yields,

$$\int_{B_\delta} |\nabla u|^2 dx = \int_{\partial B_\delta} \frac{\partial u}{\partial n} \cdot u dx + \int_{B_\delta} f(u) udx. \quad (7)$$
Thus
\[ \int_{\Omega} (\Delta u + f(u)) \cdot (\nabla u \cdot x) \, dx = \int_{\Omega} \frac{\delta^2}{\delta u^2} \left( \frac{\delta}{\delta m} u \right)^2 + \frac{\delta}{\delta u} \cdot u + F(u) \, ds + \frac{N}{2} \int_{\partial \Omega} f(u) u - 2F(u) \, ds. \tag{8} \]

Note that
\[ f(u)u - NF(u) = \left( u^{m+2} \left( 1 - \frac{N}{m + 2} \right) - u^{m+1} \alpha \left( 1 - \frac{N}{m + 1} \right) + u^2 \beta \left( 1 - \frac{N}{2} \right) \right) \]
and for \( N = 2 \),
\[ f(u)u - NF(u) = u^{m+1} \left( \frac{mu}{m + 2} - \alpha(m - 1) \right). \tag{9} \]

References


